NORMAL DEPTH EQUATION FOR VISCOUS/LAMINAR FLOW IN A RECTANGULAR CHANNEL SECTION

by

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ABSTRACT

Rectangular open channel section is adopted in chemical plants for transferring viscous fluids. Normal depth is an important parameter occurring in the design of such channel section. Direct analytic solution of normal-depth problems is not possible, as the governing equation is implicit. The solution requires tedious methods of trial and error. Reported herein are explicit equations for normal depth for viscous flow in rectangular channel section.

KEYWORDS: Channel design, Normal depth, Explicit equations, Uniform flow, and Viscous flow.

INTRODUCTION

Open channel sections are generally adopted in chemical plants for transferring viscous fluids. The type of flow in such cases is generally uniform flow. The depth of flow corresponding to uniform flow is known as the normal depth. The normal depth is a key parameter occurring in the design of channels. Also, for the analysis of nonuniform flow, normal depth is an important parameter. A review of literature reveals that a number of methods for the normal depth computation are available for turbulent flow. Chow (1973) provided a graphic procedure for the direct solution of the normal depth in rectangular and trapezoidal channels and in circular conduits running partially full. Graphic solutions were also presented by Jeppson (1965) for particular channel geometric shapes. Babaeyan-Koopaei (2001) presented graphs for round-bottomed triangular, parabolic, and round-corner rectangular cross sections. Swamee (1994) proposed explicit equations for normal depth in rectangular, trapezoidal, triangular, and circular sections using general resistance equation. Shrestha and Barkdoll (2005) obtained a regressed equation to get normal depth directly for a trapezoidal section using Manning equation. Attari and Foroghi (1995) also presented

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some explicit equations for these common cross sections. Shirley and Lopes (1991) proposed a computer program for quickly and accurately finding the unique solution, using the Chezy or Manning flow resistance equations. Golding (1996) and Montes (1996) also presented computer programs for computing normal depths. Using the Manning equation, Barr and Das (1986) presented a numerical solution for rectangular channels and both numerical and graphic procedures for trapezoidal channels and circular conduits running partially full. When the channel cross section is the unknown, the solution generally cannot be found explicitly, and for some types of channel cross section the problem does not always have a unique solution. For example, sufficiently high flow in a circular conduit will not have a unique solution (Barr and Das 1986). Wong and Zhou (2004) used Excel sheet to compute normal depth in triangular, trapezoidal, parabolic and circular cross sections. Swamee and Rathie (2004) obtained exact analytical solution for normal depth. Swamee (1995), Swamee et al. (2000), Swamee and Swamee (2008) etc. have also used normal depth as a design parameter in channels.

Swamee (2000) obtained explicit equations for the optimal dimensions of various open channel sections carrying viscous fluids. From the above review it can be seen that no such methods are available for computation of the normal depth for viscous flow in a rectangular channel. In this paper an explicit equation for the computation of the normal depth for a rectangular channel carrying viscous fluids has been obtained and its straightforwardness in application has been pointed out through a worked out example.

ANALYTICAL CONSIDERATIONS

The Navier-Stokes (N-S) equations are the governing equations for viscous flow. For a channel flow, the N-S equation for incompressible Newtonian fluid in $x$-direction is

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = X - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 v_x$$

(1)

where $x$, $y$ and $z = \text{coordinate directions}; v_x, v_y$ and $v_z = \text{velocity components in the coordinate directions}; X = \text{body force per unit mass in } x \text{ direction}; p = \text{pressure}; \rho = \text{mass density of fluid}; \text{and } \nu = \text{kinematic viscosity of fluid}. \text{The kinematic viscosity depends on the temperature of the fluid. For water it can be read from Table 1 (Streeter and Wylie 1979) or can be obtained using equation given by Swamee (2004)}$

$$\nu = 1.792 \times 10^{-6} \left[ 1 + \left( \frac{T}{25} \right)^{1.165} \right]^{-1}$$

(2)
where $T =$ the water temperature in degree Celsius.

If the $x$-coordinate direction is assumed along the channel bed in the flow direction, then $X = g S_o$, wherein $g =$ gravitational acceleration; and $S_o =$ longitudinal channel bed slope, which is equal to energy slope in uniform flow. Considering steady state uniform motion in $x$-direction and neglecting inertial terms and variation in $x$-direction, Eq. (1) reduces to

$$0 = g S_o + v \nabla^2 v_x$$

Rewriting Eq. (3)

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = - \frac{g S_o}{v}$$

where $v = v_x$ is time mean velocity in $x$ direction at the point $(y, z)$. Thus for steady viscous uniform flow, the Navier-Stokes equation is reduced to two dimensional form of Poisson’s equation as given by Eq. (4). The solution of Eq. (4) for discharge $Q$ in an open rectangular channel as derived by Boussinesq in 1868 and later modified by Cornish (1928) and Woo and Brater (1961) is

$$Q = \frac{g S_o b y_n^3}{3v} \left[ 1 - \frac{384}{\pi^3} \frac{y_n}{b} \sum_{n=0}^{\infty} \frac{(2n+1)^{-5}}{(2n+1)^{n+1}} \tanh \left( \frac{2n+1}{4} \frac{\pi b}{y_n} \right) \right]$$

where $b =$ the bed width of channel and $y_n =$ normal depth of flow in the channel. Eq. (5) was experimentally verified by Cornish (1928), Davis and White (1928), Straub et al. (1958), Woo and Brater (1961) and Woener et al. (1968).

The determination of normal depth for a rectangular channel section involves solution of an implicit equation i.e. Eq. (5). The solution of the implicit equations requires tedious method of trial and error. The Newton-Raphson method has been the usual numerical technique for solving the implicit problem of determining normal flow depth in a computer (McLatchy 1989). However, the method is sensitive (time-wise) to starting position and, for some problems, there is no guarantee that the method will converge to a unique solution (Press et al. 1986). Equation (5) can be written as

$$\frac{vQ}{g S_o b^3} = \frac{y_n^3}{3b^3} \left[ 1 - \frac{384}{\pi^3} \frac{y_n}{b} \sum_{n=0}^{\infty} \frac{(2n+1)^{-5}}{(2n+1)^{n+1}} \tanh \left( \frac{\pi}{4} \frac{(2n+1)}{y_n/b} \right) \right]$$

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where $x$, $y$, and $z$ = coordinate directions; $v_x$, $v_y$, and $v_z$ = velocity components in the coordinate directions; $X$ = body force per unit mass in $x$ direction; $p$ = pressure; $\rho$ = mass density of fluid; and $\nu$ = kinematic viscosity of fluid. The kinematic viscosity depends on the temperature of the fluid. For water it can be read form Table 1 (Streeter and Wylie 1979) or can be obtained using equation given by Swamee (2004)

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$$0 = gS_o + v \nabla^2 v_x$$  \hspace{1cm} (3)

Rewriting Eq. (3)

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where $v = v_x$ is time mean velocity in $x$ direction at the point $(y, z)$. Thus for steady viscous uniform flow, the Navier-Stokes equation is reduced to two dimensional form of Poisson's equation as given by Eq. (4). The solution of Eq. (4) for discharge $Q$ in an open rectangular channel as derived by Boussinesq in 1868 and later modified by Cornish (1928) and Woo and Brater (1961) is

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where $b$ = the bed width of channel and $y_n$ = normal depth of flow in the channel. Eq. (5) was experimentally verified by Cornish (1928), Davis and White (1928), Straub et al. (1958), Woo and Brater (1961) and Woener et al. (1968).

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$$\frac{vQ}{gS_o b^4} = \frac{y_n^3}{3b^3} \left[ 1 - \frac{384}{\pi^5} \frac{y_n}{b} \sum_{n=0}^{\infty} (2n+1)^{-5} \tanh \left( \frac{\pi}{4} \frac{(2n+1)}{y_n/b} \right) \right]$$  \hspace{1cm} (6)

Defining
\[ M_b = \frac{vQ}{gS_o b^4} \quad \beta_n = \frac{y_n}{b} \]  

Equation (6) becomes

\[ M_b = \frac{\beta_n^3}{3} \left[ 1 - \frac{384}{\pi^5} \beta_n \sum_{n=0}^{\infty} (2n+1)^{-5} \tanh \left( \frac{\pi}{4} \frac{2n+1}{\beta_n} \right) \right] \]  

Equation (8) is implicit in \( \beta_n \) therefore it is not convenient for determination of normal depth \( y_n \). The firm line in graph (Fig. 1) shows the variation in \( \beta_n \) with \( M_b \). Following the procedure similar to the diameter equation investigated by Swamee and Jain (1976), an explicit equation for \( \beta_n \) for a rectangular section has been obtained. The fitted equation is

\[ \beta_n = 1.55 M_b^{0.337} \frac{1+10.01 M_b^{0.881}}{1+0.391 M_b^{0.645}} \]  

FIG 1 VARIATION IN \( \beta_n \) WITH \( M_b \)
FIG. 2. ERROR IN FITTED EQUATION

The dotted line in graph (Fig. 1) shows the variation in computed $\beta_n$ using Eq. (9) with $M_b$. Both the graphs (actual and computed) almost overlap each other and are indistinguishable in the range of $\beta_n$ from 0.01 to 25. The associated error in this range is less than 2.5% as shown in Fig. 2. Thus Eq. (9) is sufficiently accurate for all practical purposes. Rewriting Eq. (9)

$$y_n = 1.55bM_b^{0.337} \frac{1+10.01M_b^{0.881}}{1+0.391M_b^{0.645}}$$ (10)

For given parameters (discharge, bed width, bed slope and viscosity of fluid), the dimensionless parameter $M_b$ can be computed using Eq. (7a) and then Eq. (9) or (10) results the normal depth in single step computation involving a negligible error.

It should be noted that Eq. (5) and hence Eqs. (6), (8), (9) and (10) are applicable in the laminar flow range only. The upper limit of Reynolds number for laminar flow in open channel is 500, therefore the range of validity of the proposed equations are

$$\frac{Vby_n}{\nu(b+2y_n)} < 500$$ (11)
EXAMPLE

Determine the normal depth in 0.5 m wide rectangular channel section for carrying a discharge of 0.025 m$^3$/s on a bed slope of 0.005. The kinematic viscosity of fluid is $4 \times 10^{-3}$ m$^2$/s.

Adopting $g = 9.79$ m/s$^2$ and then using Eq. (7a); $M_b = 0.0327$ m. Solving Eq. (8) by Trial and error method; $\beta_n = 0.6997$, hence $y_n = 0.3499$ m; while Eq. (9) directly yields $\beta_n = 0.6998$ or $y_n = 0.3499$ m. Thus the error in the normal depth computation = -0.02 %, which is negligible.

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \times 10^{-6}$ m$^2$/s</td>
<td>1.792</td>
<td>1.519</td>
<td>1.308</td>
<td>1.141</td>
<td>1.007</td>
<td>0.897</td>
<td>0.804</td>
<td>0.727</td>
</tr>
</tbody>
</table>

CONCLUSIONS

Design of open channels carrying viscous fluids requires computation of the normal depth. Direct analytical solution of normal depth in a rectangular channel carrying viscous fluid is not possible because of the implicit nature of governing equations. The solution usually requires tedious iterative techniques. Using minimization of errors between the actual values and the computed values from a properly conceived function, it has been possible to obtain an explicit equation for the normal depth in a rectangular channel carrying viscous fluid. The proposed equation is near exact and may be useful to design engineers interested in computation of normal depth for viscous flow.

REFERENCES


NOTATIONS

\[ b \] \quad \text{bed width} \\
\[ g \] \quad \text{gravitational acceleration} \\
\[ M_b \] \quad \text{dimensionless discharge parameter} \\
\[ p \] \quad \text{pressure} \\
\[ Q \] \quad \text{discharge} \\
\[ S_o \] \quad \text{bed slope} \\
\[ T \] \quad \text{temperature in degree Celsius} \\
\[ v \] \quad \text{time mean velocity in } x \text{ direction at the point } (y, z) \\
\[ v_x, v_y, v_z \] \quad \text{velocity components in the coordinate directions} \\
\[ X \] \quad \text{body force per unit mass in } x \text{ direction} \\
\[ x, y, z \] \quad \text{coordinate directions} \\
\[ y_n \] \quad \text{normal depth} \\
\[ \beta_n \] \quad \text{dimensionless normal depth parameter} \\
\[ \nu \] \quad \text{kinematic viscosity} \\
\[ \rho \] \quad \text{mass density of fluid}